International J.Math. Combin. Vol.1(2017), 36-52

Some New Generalizations of the Lucas Sequence

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Abstract: In this paper, we investigate the generalized Lucas, the generalized complex Lucas and the generalized dual Lucas sequence using the Lucas number. Also, we investigate special cases of these sequences. Furthermore, we give recurrence relations, vectors, the golden ratio and Binet's formula for the generalized Lucas and the generalized dual Lucas sequence.

Key Words: Smarandache-Fibonacci triple, Fibonacci number, Lucas number, Lucas sequence, generalized Fibonacci sequence, generalized complex Lucas sequence, generalized dual Lucas sequence.

AMS(2010): 11B37, 11B39.

§1. Introduction

Let $S(n), n \geq 0$ with $S(n) = S(n-1) + S(n-2)$ be a Smarandache-Fibonacci triple, where $S(n)$ is the Smarandache function for integers $n \geq 0$. Particularly, let $S(n)$ be $F(n)$ or $L(n)$, we get the Fibonacci or Lucas sequence as follows:

A Fibonacci sequence

$$1\,,\ 1\,,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ 144,\ 233, \cdots, F_n, \cdots$$

is defined by the recurrence relation

$$F_n = F_{n-1} + F_{n-2}\ ,\quad (n \geqslant 3),$$

with $F_1 = F_2 = 1$, where F_n is the n-th term of the Fibonacci sequence (F_n) (Leonardo Fibonacci, 1202). The Fibonacci sequence is named after Italian mathematician Leonardo of Pisa, known as Fibonacci. The name "Fibonacci Sequnce" was first used by the $19th$- century number theorist Edouard Lucas. Some recent generalizations for the Fibonacci sequence have

[1]Received May 17, 2016, Accepted February 8, 2017.

produced a variety of new and extended results, [1],[5],[6],[9],[13].

A Lucas sequence

$$2\,,\ 1\,,\ 3,\ 4,\ 7,\ 11,\ 18,\ 29,\ 47,\ 76,\ 123,\ 199,\ 322,\cdots,L_n,\cdots$$

is defined by the recurrence relation

$$L_n = L_{n-1} + L_{n-2}\ ,\quad (n \geqslant 3),$$

with $L_1 = 2\,, L_2 = 1$, where L_n is the n-th term of the Lucas sequence (L_n) (François Edouard Anatole Lucas, 1876). There are a lot of generalizations of the Lucas sequences,[15],[16],[17].

The generalized Fibonacci sequence defined by

$$H_n = H_{n-1} + H_{n-2}\ ,\quad (n \geqslant 3) \tag{1.1}$$

with $H_1 = p\,,\ H_2 = p+q$ where p, q are arbitrary integers [3]. That is, the generalized Fibonacci sequence is

$$p\,,\ p+q\,,\ 2p+q,\ 3p+2q,\ 5p+3q,\ 8p+5q,\cdots,(p-q)F_n + qF_{n+1},\cdots \tag{1.2}$$

Using the equations (1.1) and (1.2) , it was obtained

$$\begin{aligned} H_{n+1} &= q\,F_n + p\,F_{n+1} \\ H_{n+2} &= p\,F_n + (p+q)\,F_{n+1}. \end{aligned} \tag{1.3}$$

For the generalized Fibonacci sequence, it was obtained the following properties:

$$H_{n-1}^2 + H_n^2 = (2p-q)H_{2n-1} - e\,F_{2n-1}\,, \tag{1.4}$$

$$H_{n+1}^2 - H_{n-1}^2 = (2p-q)H_{2n} - e\,F_{2n}\,, \tag{1.5}$$

$$H_{n-1}\,H_{n+1} - H_n^2 = (-1)^n e\ , \tag{1.6}$$

$$H_{n+r} = H_{n-1}F_r + H_nF_{r+1} \qquad (n \geqslant 3) \tag{1.7}$$

$$H_{n+1-r}\,H_{n+1+r} - H_{n+1}^2 = (-1)^{n-r} e\ F_r^2\ , \tag{1.8}$$

$$H_{n+1}^2 + e\ F_n^2 = p\,H_{2n+1}\,, \tag{1.9}$$

$$H_n\,H_{n+1+r} - H_{n-s}\,H_{n+r+s+1} = (-1)^{n+s} e\ F_sF_{r+s+1}\ , \tag{1.10}$$

$$\left[2H_{n+1}H_{n+2}\right]^2 + \left[H_nH_{n+3}\right]^2 = \left[2H_{n+1}H_{n+2} + H_n^2\right]^2 \tag{1.11}$$

$$\frac{H_{n+r} + (-1)^r H_{n-r}}{H_n} = F_{r+1} + (-1)^r F_{r-1} \tag{1.12}$$

where e $= p^2 - p\,q - q^2$.

Also, for $p = 1, q = 0$, we get the following well-known results:

$$F_{n-1}^2 + F_n^2 = F_{2n-1}\,, \quad \text{(Catalan)}, \tag{1.13}$$

$$F_{n-1}\,F_{n+1} - F_n^2 = (-1)^n\,, \quad \text{(Simpson or Cassini)}, \tag{1.14}$$

$$F_{n+1}^2 + F_n^2 = F_{2n+1} \quad \text{(Lucas)}. \tag{1.15}$$

In this paper, we will define the generalized Lucas, the generalized complex Lucas and the generalized dual Lucas sequences respectively, denoted by G_n , $\mathbb{C}_n$, $\mathbb{D}_n$.

§2. Generalized Lucas Sequence and Lucas Vectors

In this section, we will define the generalized Lucas sequence denoted by $\mathbb{L}_n$. The generalized Lucas sequence defined by

$$\mathbb{L}_n = \mathbb{L}_{n-1} + \mathbb{L}_{n-2}\,, \quad (n \geqslant 3), \tag{2.1}$$

with $\mathbb{L}_1 = 2\,p - \,q$, $\mathbb{L}_2 = p + 2\,q$ where p, q are arbitrary integers,[3]. That is, the generalized Lucas sequence is

$$2\,p - q\,,\; p + 2\,q\,,\; 3\,p + q\,,\; 4\,p + 3\,q\,,\; 7\,p + 4\,q\,,\; 11\,p + 7\,q\,, \cdots, (p - q)L_n + qL_{n+1}, \cdots \tag{2.2}$$

Using the equations (2.1) and (2.2) , we get

$$\begin{aligned} \mathbb{L}_{n+1} &= q\,L_n + p\,L_{n+1}\,, \\ \mathbb{L}_{n+2} &= p\,L_n + (p + q)\,L_{n+1}. \end{aligned} \tag{2.3}$$

Putting $n = r$ in (2.3) and using (2.1), we find in turn

$$\begin{aligned} \mathbb{L}_{r+3} &= (2\,p + q\,)L_{r+1} + (p + q\,)\,L_r = H_3 L_{r+1} + H_2 L_r \\ \mathbb{L}_{r+4} &= (3\,p + 2\,q)\,L_{r+1} + (2\,p + q)\,L_r = H_4 L_{r+1} + H_3 L_r \end{aligned} \tag{2.4}$$

So, in general, we have obtain relations between generalized Lucas sequence and generalized Fibonacci sequence as follows:

$$\mathbb{L}_{n+r} = H_{n-1} L_r + H_n L_{r+1} \tag{2.5}$$

Also, certain results follow almost immediately from (2.1)

$$\mathbb{L}_{n+2} - 2\,\mathbb{L}_n - \mathbb{L}_{n-1} = 0, \tag{2.6}$$

$$\mathbb{L}_{n+1} - 2\,\mathbb{L}_n + \mathbb{L}_{n-2} = 0, \tag{2.7}$$

$$\sum_{i=0}^{n-1} \mathbb{L}_{2i+1} = \mathbb{L}_{2n} - (2\,p - q), \tag{2.8}$$

$$\sum_{i=1}^{n} \mathbb{L}_{2i} = \mathbb{L}_{2n+1} - (p + 2\,q), \tag{2.9}$$

$$\sum_{i=1}^{n} (\mathbb{L}_{2i-1} - \mathbb{L}_{2i}) = -\,\mathbb{L}_{2n-1} - p + 3\,q. \tag{2.10}$$

For the generalized Lucas sequence, we have the following properties:

$$\mathbb{L}_{n-1}^2 + \mathbb{L}_n^2 = (2p - q)(\mathbb{L}_{2n-2} + \mathbb{L}_{2n}) - e_L\left(L_{2n-2} + L_{2n}\right), \tag{2.11}$$

$$\mathbb{L}_{n+1}^2 - \mathbb{L}_{n-1}^2 = (2p - q)(\mathbb{L}_{2n+2} - \mathbb{L}_{2n-2}) - e_L\left(L_{2n+2} - L_{2n-2}\right), \tag{2.12}$$

$$\mathbb{L}_{n-1}\,\mathbb{L}_{n+1} - \mathbb{L}_n^2 = 5\,(-1)^{n+1}\,e_L\ , \tag{2.13}$$

$$\mathbb{L}_{n+1}^2 + e_L\ L_n^2 = p\left(\mathbb{L}_{2n+2} + \mathbb{L}_{2n}\right), \tag{2.14}$$

$$\frac{\mathbb{L}_{n+r} + (-1)^r \mathbb{L}_{n-r}}{\mathbb{L}_n} = L_r \tag{2.15}$$

where $e_L = p^2 - p\,q - q^2$.

Theorem 2.1 *If $\mathbb{L}_n$ is the generalized Lucas number, then*

$$\lim_{n\to\infty} \frac{\mathbb{L}_{n+1}}{\mathbb{L}_n} = \frac{p\alpha + q}{q\alpha + (p - q)},$$

where $\alpha = (1+\sqrt{5})/2 = 1.618033\cdots$ is the golden ratio.

Proof We have for the Lucas number L_n,

$$\lim_{n\to\infty} \frac{L_{n+1}}{L_n} = \alpha,$$

where

$$\alpha = (1+\sqrt{5})\Big/2 = 1.618033\cdots$$

is the golden ratio [12].

Then for the generalized Lucas number $\mathbb{L}_n$, we obtain

$$\lim_{n\to\infty} \frac{\mathbb{L}_{n+1}}{\mathbb{L}_n} = \lim_{n\to\infty} \frac{pL_{n+1} + qL_n}{qL_{n+1} + (p-q)L_n} = \frac{p\alpha + q}{q\alpha + (p - q)}. \tag{2.16}$$

$\square$

Theorem 2.2 *The Binet's formula*[2] *for the generalized Lucas sequence is as follows;*

$$\mathbb{L}_n = (\,\bar{\alpha}\ \alpha^n + \bar{\beta}\ \beta^n) \tag{2.17}$$

where $\bar{\alpha} = \alpha(2\,p - q) - (p + 2\,q)$, $\bar{\beta} = (p + 2\,q) - \beta\,(2\,p - q)$.

Proof The characteristic equation of recurrence relation $\mathbb{L}_{n+2} = \mathbb{L}_{n+1} + \mathbb{L}_n$ is

$$t^2 - t - 1 = 0. \tag{2.18}$$

The roots of this equation are

$$\alpha = \frac{1+\sqrt{5}}{2} \text{ and } \beta = \frac{1-\sqrt{5}}{2}, \tag{2.19}$$

where $\alpha + \beta = 1$, $\alpha - \beta = \sqrt{5}$, $\alpha\beta = -1$.

Using recurrence relation and initial values $\mathbb{L}_0 = (2\,p - q)$, $\mathbb{L}_1 = (p + 2\,q)$ the Binet's formula for $\mathbb{L}_n$, we get

$$\mathbb{L}_n = A\ \alpha^n + B\,\beta^n = \left[\,\bar{\alpha}\ \alpha^n + \bar{\beta}\ \beta^n\right], \tag{2.20}$$

where

$$A = \frac{\mathbb{L}_1 - \beta\,\mathbb{L}_0}{\alpha - \beta},\ B = \frac{\alpha\,\mathbb{L}_0 - \mathbb{L}_1}{\alpha - \beta}$$

and $\bar{\alpha} = \alpha(2\,p - q) - (p + 2\,q)$, $\bar{\beta} = (p + 2\,q) - \beta\,(2\,p - q)$. □

A generalized Lucas vector is defined by

$$\overrightarrow{\mathbb{L}_{\mathrm{n}}} = (\mathbb{L}_n\,,\,\mathbb{L}_{n+1},\,\mathbb{L}_{n+2})$$

Also, from equation (2.2) it can be expressed as

$$\overrightarrow{\mathbb{L}_{\mathrm{n}}} \quad = (p-q)\,\overrightarrow{\mathrm{L}}_n + q\,\overrightarrow{\mathrm{L}}_{\mathrm{n+1}} \tag{2.21}$$

where $\overrightarrow{\mathrm{L}}_{\mathrm{n}} = (\,L_n\,,\,L_{n+1},\,L_{n+2})$ and $\overrightarrow{\mathrm{L}}_{\mathrm{n+1}} = (\,L_{n+1}\,,\,L_{n+2},\,L_{n+3})$ are the Lucas vectors.

The product of $\overrightarrow{\mathbb{L}_{\mathrm{n}}}$ and $\lambda \in \mathbb{R}$ is given by

$$\lambda\,\overrightarrow{\mathbb{L}_{\mathrm{n}}} = (\lambda\,\overrightarrow{\mathbb{L}_{\mathrm{n}}},\,\lambda\,\overrightarrow{\mathbb{L}_{\mathrm{n+1}}},\,\lambda\,\overrightarrow{\mathbb{L}_{\mathrm{n+2}}})$$

[2]Binet's formula is the explicit formula to obtain the n-th Fibonacci and Lucas numbers. It is well known that for the Fibonacci and Lucas numbers, Binet's formulas are

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

and

$$L_n = \alpha^n + \beta^n$$

respectively, where $\alpha + \beta = 1$, $\alpha - \beta = \sqrt{5}$, $\alpha\beta = -1$ and $\alpha = (1+\sqrt{5})\big/2$, $\beta = (1-\sqrt{5})\big/2$,[7], [8].

and $\overrightarrow{\mathbb{L}_\mathrm{n}}$ and $\overrightarrow{\mathbb{L}_\mathrm{m}}$ are equal if and only if

$$\begin{aligned} \mathbb{L}_n &= \mathbb{L}_m \\ \mathbb{L}_{n+1} &= \mathbb{L}_{m+1} \\ \mathbb{L}_{n+2} &= \mathbb{L}_{m+2}. \end{aligned}$$

Theorem 2.3 *Let* $\overrightarrow{\mathbb{L}_\mathrm{n}}$ *and* $\overrightarrow{\mathbb{L}_\mathrm{m}}$ *be two generalized Lucas vectors. The dot product of* $\overrightarrow{\mathbb{L}_\mathrm{n}}$ *and* $\overrightarrow{\mathbb{L}_\mathrm{m}}$ *is given by*

$$\begin{aligned} \left\langle \overrightarrow{\mathbb{L}_\mathrm{n}}, \overrightarrow{\mathbb{L}_\mathrm{m}} \right\rangle &= p^2\,(5\,F_{n+m+3} + L_n\,L_m) \\ &\quad + p\,q\,[2\,L_{n+m-1} + 10\,F_{n+m+2}\,] \\ &\quad + q^2 (5\,F_{n+m+1} + L_{n-1} L_{m-1})\,. \end{aligned} \tag{2.22}$$

Proof The dot product of $\overrightarrow{\mathbb{L}_\mathrm{n}} = (\mathbb{L}_n, \mathbb{L}_{n+1}, \mathbb{L}_{n+2})$ and $\overrightarrow{\mathbb{L}_\mathrm{m}} = (\mathbb{L}_m, \mathbb{L}_{m+1}, \mathbb{L}_{m+2})$ defined by

$$\left\langle \overrightarrow{\mathbb{L}_\mathrm{n}}, \overrightarrow{\mathbb{L}_\mathrm{m}} \right\rangle = \mathbb{L}_n \mathbb{L}_m + \mathbb{L}_{n+1}\mathbb{L}_{m+1} + \mathbb{L}_{n+2}\mathbb{L}_{m+2}.$$

Also, using the equations (2.1), (2.2) and (2.3), we obtain

$$\mathbb{L}_n\,\mathbb{L}_m = p^2\,(L_n\,L_m) + p\,q\,[L_n L_{m-1} + L_{n-1}\,L_m] + q^2\,(L_{n-1}\,L_{m-1}), \tag{2.23}$$

$$\mathbb{L}_{n+1}\,\mathbb{L}_{m+1} = p^2\,(L_{n+1}\,L_{m+1}) + p\,q\,[L_{n+1} L_m + L_n\,L_{m+1}] + q^2\,(L_n\,L_m), \tag{2.24}$$

$$\begin{aligned} \mathbb{L}_{n+2}\,\mathbb{L}_{m+2} &= p^2\,(L_{n+2} L_{m+2}) + p\,q\,[L_{n+2} L_{m+1} + L_{n+1} L_{m+2}] \\ &\quad + q^2\,(L_{n+1} L_{m+1}). \end{aligned} \tag{2.25}$$

Then, from the equations (2.23), (2.24) and (2.25), we have

$$\begin{aligned} \left\langle \overrightarrow{\mathbb{L}_\mathrm{n}}, \overrightarrow{\mathbb{L}_\mathrm{m}} \right\rangle &= p^2 (L_n L_m + L_{n+1} L_{m+1} + L_{n+2} L_{m+2}) \\ &\quad + (p\,q)\,[L_n L_{m-1} + L_{n-1} L_m + L_{n+1} L_m + L_n L_{m+1} \\ &\quad + L_{n+2} L_{m+1} + L_{n+1} L_{m+2}] \\ &\quad + q^2\,(L_{n-1}\,L_{m-1} + L_n\,L_m + L_{n+1}\,L_{m+1}) \\ &= p^2 (5\,F_{n+m+3} + L_n L_m) \\ &\quad + (p\,q)\,[10 F_{n+m+2} + 2\,L_{n+m-1}] \\ &\quad + q^2 (5\,F_{n+m+1} + L_{n-1}\,L_{m-1})\,. \end{aligned} \tag{2.26}$$

$\square$

Case 1. For the dot product of the generalized Lucas vectors $\overrightarrow{\mathbb{L}_{\mathrm{n}}}$ and $\overrightarrow{\mathbb{L}}_{\mathrm{n+1}}$, we get

$$\begin{aligned}\left\langle \overrightarrow{\mathbb{L}_{\mathrm{n}}}, \overrightarrow{\mathbb{L}_{\mathrm{n+1}}} \right\rangle &= \mathbb{L}_n \mathbb{L}_{n+1} + \mathbb{L}_{n+1}\mathbb{L}_{n+2} + \mathbb{L}_{n+2}\mathbb{L}_{n+3} \\ &= p^2\,[\,5\,F_{2n+4} + L_n L_{n+1}] \\ &\quad +(p\,q)\,[\,10\,F_{2n+3} + 2\,L_{2n}\,] \\ &\quad +q^2\,[\,5\,F_{2n+2} + L_{n-1}\,L_n]\end{aligned} \tag{2.27}$$

and

$$\begin{aligned}\left\langle \overrightarrow{\mathbb{L}_{\mathrm{n}}}, \overrightarrow{\mathbb{L}_{\mathrm{n}}} \right\rangle &= (\mathbb{L}_n)^2 + (\mathbb{L}_{n+1})^2 + (\mathbb{L}_{n+2})^2 \\ &= p^2[\,L_n^2 + L_{n+1}^2 + L_{n+2}^2] \\ &\quad +(p\,q)[\,2\,L_n L_{n-1} + 2\,L_{n+1}L_n + 2\,L_{n+2}L_{n+1}] \\ &\quad +q^2[\,L_{n-1}^2 + L_n^2 + L_{n+1}^2].\end{aligned} \tag{2.28}$$

Then for the norm of the generalized Lucas vector, using identities of the Fibonacci numbers

$$\begin{aligned} L_{n+1}^2 + L_n^2 &= 5\,F_{2n+1} \\ L_{n+1}^2 - L_{n-1}^2 &= 5\,F_{2n} \\ L_{n+1}^2 - L_n^2 &= L_{n-1}\,L_{n+2} \\ L_n L_m + L_{n+1}L_{m+1} &= 5\,F_{n+m+1} \end{aligned}$$

we have

$$\begin{aligned}\left\| \overrightarrow{\mathbb{L}_{\mathrm{n}}} \right\|^2 &= \left\langle \overrightarrow{\mathbb{L}_{\mathrm{n}}}, \overrightarrow{\mathbb{L}_{\mathrm{n}}} \right\rangle = {\mathbb{L}_n}^2 + \mathbb{L}_{n+1}^2 + \mathbb{L}_{n+2}^2 \\ &= p^2\left[5\,F_{2n+3} + L_n^2\right] \\ &\quad +(p\,q)\left[2\,F_{2n+2} + 2\,L_n\,L_{n-1}\right] \\ &\quad +q^2\left[5\,F_{2n+1} + L_{n-1}^2\right].\end{aligned} \tag{2.29}$$

Case 2. For $p = 1$, $q = 0$, in the equations (2.26), (2.27) and (2.29), we have

$$\begin{aligned}\left\langle \overrightarrow{\mathbb{L}_{\mathrm{n}}}, \overrightarrow{\mathbb{L}_{\mathrm{m}}} \right\rangle &= [\,5\,F_{n+m+3} + L_n\,L_m]\,, \\ \left\langle \overrightarrow{\mathbb{L}_{\mathrm{n}}}, \overrightarrow{\mathbb{L}_{\mathrm{n+1}}} \right\rangle &= [\,5\,F_{2n+4} + L_n\,L_{n+1}]\end{aligned}$$

and

$$\left\| \overrightarrow{\mathbb{L}_{\mathrm{n}}} \right\| = \sqrt{5\,F_{2n+3} + L_n^2}\,.$$

Theorem 2.4 *Let $\overrightarrow{\mathbb{L}_{\mathrm{n}}}$ and $\overrightarrow{\mathbb{L}_{\mathrm{m}}}$ be two generalized Lucas vectors. The cross product of $\overrightarrow{\mathbb{L}_{\mathrm{n}}}$ and $\overrightarrow{\mathbb{L}_{\mathrm{m}}}$ is given by*

$$\overrightarrow{\mathbb{L}_{\mathrm{n}}} \times \overrightarrow{\mathbb{L}_{\mathrm{m}}} = 5\,(-1)^n\,F_{m-n}\,(p^2 - p\,q - q^2)\,(i + j - k). \tag{2.30}$$

Proof The cross product of $\overrightarrow{\mathbb{L}_{\rm n}} \times \overrightarrow{\mathbb{L}_{\rm m}}$ defined by

$$\overrightarrow{\mathbb{L}_{\rm n}} \times \overrightarrow{\mathbb{L}_{\rm m}} = \begin{vmatrix} i & j & k \\ \mathbb{L}_n & \mathbb{L}_{n+1} & \mathbb{L}_{n+2} \\ \mathbb{L}_m & \mathbb{L}_{m+1} & \mathbb{L}_{m+2} \end{vmatrix} \tag{2.31}$$

$$= i\left(\mathbb{L}_{n+1}\mathbb{L}_{m+2} - \mathbb{L}_{n+2}\mathbb{L}_{m+1}\right) + j\left(\mathbb{L}_{n+2}\mathbb{L}_m - \mathbb{L}_n\mathbb{L}_{m+2}\right) + k\left(\mathbb{L}_n\mathbb{L}_{m+1} - \mathbb{L}_{n+1}\mathbb{L}_m\right).$$

Now, we calculate the cross products. Using the property $L_nL_{m+1} - L_{n+1}L_m = 5\,(-1)^n\,F_{m-n}$ we get

$$\mathbb{L}_{n+1}\,\mathbb{L}_{m+2} - \mathbb{L}_{n+2}\,\mathbb{L}_{m+1} = 5\,(-1)^n F_{m-n}\,(p^2 - p\,q - q^2) = 5\,(-1)^n F_{m-n}\,e_L, \tag{2.32}$$

$$\mathbb{L}_{n+2}\,\mathbb{L}_m - \mathbb{L}_n\,\mathbb{L}_{m+2} = 5\,(-1)^n F_{m-n}\,(p^2 - pq - q^2) = 5\,(-1)^n F_{m-n}\,e_L, \tag{2.33}$$

and

$$\mathbb{L}_n\,\mathbb{L}_{m+1} - \mathbb{L}_{n+1}\,\mathbb{L}_m = 5\,(-1)^{n+1} F_{m-n}\,(p^2 - p\,q - q^2) = 5\,(-1)^{n+1} F_{m-n}\,e_L\,. \tag{2.34}$$

Then from the equations (2.32), (2.33) and (2.34), we obtain the equation (2.30).

Case 3. For $p = 1$, $q = 0$, in the equation (2.30), we have

$$\overrightarrow{\mathbb{L}_{\rm n}} \times \overrightarrow{\mathbb{L}_{\rm m}} = 5\,(-1)^n F_{m-n}\,(i + j - k).$$

□

Theorem 2.5 *Let $\overrightarrow{\mathbb{L}_{\rm n}}$, $\overrightarrow{\mathbb{L}_{\rm m}}$ and $\overrightarrow{\mathbb{L}_{\rm k}}$ be the generalized Lucas vectors. The mixed product of these vectors is*

$$\left\langle \overrightarrow{\mathbb{L}_{\rm n}} \times \overrightarrow{\mathbb{L}_{\rm m}}\,,\; \overrightarrow{\mathbb{L}_{\rm k}} \right\rangle = 0. \tag{2.35}$$

Proof Using $\overrightarrow{\mathbb{L}_{\rm k}} = (\mathbb{L}_k, \mathbb{L}_{k+1}, \mathbb{L}_{k+2})$, we can write,

$$\left\langle \overrightarrow{\mathbb{L}_{\rm n}} \times \overrightarrow{\mathbb{L}_{\rm m}}\,,\; \overrightarrow{\mathbb{L}_{\rm k}} \right\rangle = \begin{vmatrix} \mathbb{L}_n & \mathbb{L}_{n+1} & \mathbb{L}_{n+2} \\ \mathbb{L}_m & \mathbb{L}_{m+1} & \mathbb{L}_{m+2} \\ \mathbb{L}_k & \mathbb{L}_{k+1} & \mathbb{L}_{k+2} \end{vmatrix} \tag{2.36}$$

$$= \mathbb{L}_n\left(\mathbb{L}_{m+1}\,\mathbb{L}_{k+2} - \mathbb{L}_{m+2}\,\mathbb{L}_{k+1}\right) + \mathbb{L}_{n+1}\left(\mathbb{L}_{m+2}\,\mathbb{L}_k - \mathbb{L}_m\,\mathbb{L}_{k+2}\right) + \mathbb{L}_{n+2}\left(\mathbb{L}_m\,\mathbb{L}_{k+1} - \mathbb{L}_{m+1}\,\mathbb{L}_k\right).$$

Also, using the equations (2.32), (2.33) and (2.34), we obtain

$$\begin{aligned}\mathbb{L}_n\,(\mathbb{L}_{m+1}\,\mathbb{L}_{k+2}-\mathbb{L}_{m+2}\,\mathbb{L}_{k+1}) \quad & +\mathbb{L}_{n+1}\,(\mathbb{L}_{m+2}\,\mathbb{L}_k-\mathbb{L}_m\,\mathbb{L}_{k+2}) \\ & +\mathbb{L}_{n+2}\,(\mathbb{L}_m\,\mathbb{L}_{k+1}-\mathbb{L}_k\,\mathbb{L}_{m+1}) \\ = \quad & 5\,(-1)^m\,F_{k-m}\,e_L\,(\,\mathbb{L}_n+\mathbb{L}_{n+1}-\mathbb{L}_{n+2}\,) \\ = \quad & 5\,(-1)^m\,F_{k-m}\,e_L\,(\,\mathbb{L}_{n+2}-\mathbb{L}_{n+2}\,)=0.\end{aligned} \tag{2.37}$$

Thus, we have the equation (2.35). □

§3. Generalized Complex Lucas Sequence

In this section, we will define the generalized complex Lucas sequence denoted by $\mathbb{C}_n$. The generalized complex Lucas sequence defined by

$$\mathbb{C}_n=\mathbb{L}_n+i\,\mathbb{L}_{n+1}\ , \tag{3.1}$$

with $\mathbb{C}_0=(2\,p-q)\,+i\,(p+2\,q)$, $\mathbb{C}_1=(p+2\,q)\,+\,i\,(3\,p+q)$, $\mathbb{C}_2=(3\,p+q)\,+\,i\,(4\,p+3\,q)$, where p,q are arbitrary integers. That is, the generalized complex Lucas sequence is

$$\begin{gathered}(2\,p-q)\,+i\,(p+2\,q)\,,\ (p+2\,q)\,+\,i\,(3\,p+q),\ (3\,p+q)\,+i\,(4\,p+3\,q)\,, \\ (4\,p+3\,q)\,+\,i\,(7\,p+4\,q)\,,\,\ldots\,,\,(\,p-q\,+i\,q\,)L_n+\,(\,q+\,i\,p\,)L_{n+1},\ldots\end{gathered} \tag{3.2}$$

Case 1. From the generalized complex Lucas sequence $(\mathbb{C}_n)$ for $p=1,\ q=0$ in the equation (3.2), we obtain complex Lucas sequence (C_n) as follows:

$$(C_n)\ :\ 2\,+\,i\,,\ 1+i\,3\,,\ 3+i\,4\,,\ 4+i\,7,\cdots,\,L_n+\,i\,L_{n+1},\cdots.$$

For the generalized complex Lucas sequence, we have the following properties:

$$\begin{aligned}\mathbb{C}_n^2\,+\,\mathbb{C}_{n-1}^2\quad & =[\,(2p-\,q)+i\,(p\,+2\,q)\,]\,(\mathbb{C}_{2\,n-2}+\mathbb{C}_{2\,n}\,) \\ & -(\,2\,+i\,)\,e_L\,(L_{2\,n-2}+L_{2\,n}\,)\,,\end{aligned} \tag{3.3}$$

$$\begin{aligned}\mathbb{C}_{n+1}^2\,+\,\mathbb{C}_{n-1}^2\quad & =[\,(2p-\,q)+i\,(p\,+2\,q)\,]\,(\mathbb{C}_{2\,n+2}+\mathbb{C}_{2\,n-2}\,) \\ & -(\,2\,+i\,)\,e_L\,(L_{2\,n+2}+L_{2\,n-2}\,)\,,\end{aligned} \tag{3.4}$$

$$\mathbb{C}_{n-1}\,\mathbb{C}_{n+1}-\mathbb{C}_n^2=5\,(-1)^{n+1}\,(\,2\,+\,i\,)\;e_L\ , \tag{3.5}$$

$$\mathbb{C}_{n+1}^2\,+(\,2\,+i\,)\,e_L\,L_n^2\,=[\,(\,2\,p+q\,)+(\mathbb{C}_{2\,n+2}+\mathbb{C}_{2\,n}\,)\ , \tag{3.6}$$

$$\frac{\mathbb{C}_{n+r}+(-1)^r\mathbb{C}_{n-r}}{C_n}=L_r\ . \tag{3.7}$$

where $e_{\mathbb{C}}=(\,2\,+i\,)\,e_L$.

§4. Generalized Dual Lucas Sequence

In this section, we will define the generalized dual Lucas sequence denoted by $\mathbb{D}_n^L$. The generalized dual Lucas sequence defined by

$$\mathbb{D}_n^L=\mathbb{L}_n+\varepsilon\,\mathbb{L}_{n+1}\ , \tag{4.1}$$

with $\mathbb{D}_0^L=(2\,p-q)+\varepsilon\,(\,p+2\,q)\,,\ \mathbb{D}_1^L=(p+2\,q)+\varepsilon\,(3\,p+q)$ where p,q are arbitrary integers. That is, the generalized dual Lucas sequence is

$$\begin{aligned}&(2\,p-q)\ +\varepsilon\,(3\,p+q)\,,\ (p+2\,q)\ +\ \varepsilon\,(3\,p+q),\ (3\,p+q)\ +\varepsilon\,(4\,p+3\,q),\\&(\,4\,p+3\,q\,)+\varepsilon(7\,p+4\;q),\ (\,7\,p+4\,q\,)+\ \varepsilon(11\,p+7\,q\,),\\&\cdots,\ (\,p-q+\ \varepsilon\,q\,)L_n+\ (\ q+\ \varepsilon p\,)\,L_{n+1},\cdots\end{aligned} \tag{4.2}$$

Using the equations (4.1) and (4.2) , we get

$$\begin{aligned}\mathbb{D}_n^L&=(\,p-q+\ \varepsilon\,q\,)\,L_n+\ (\ q+\varepsilon\,p\,)\,L_{n+1}\,,\\ \mathbb{D}_{n+1}^L&=\ (\ q+\varepsilon\,p\,)\,L_n+[\,p\ +\ \varepsilon\,(\,p+\ q\,)\,]\,L_{n+1}\,,\\ \mathbb{D}_{n+2}^L&=\ [\,p+\varepsilon(\,p+\ q\,)\,]\,L_n+[\,(p+\ q)+\varepsilon\,(\,2\,p+\ q\,)\,]L_{n+1}\,.\end{aligned} \tag{4.3}$$

Case 1. From the generalized dual Lucas sequence ($\mathbb{D}_n^L$) for $p=1,\ \ q=0$ in the equation (4.2), we obtain dual Lucas sequence (D_n^L) as follows:

$$(D_n^L)\ :\ 2+\varepsilon\,,\,1+3\,\varepsilon\,,\ 3+4\,\varepsilon\,,\ 4+7\,\varepsilon\,,\ 7+11\,\varepsilon,\ 11+18\,\varepsilon,\cdots,\ L_n+\ \varepsilon\,L_{n+1},\cdots$$

For the generalized dual Lucas sequence, we have the following properties:

$$\begin{aligned}(\mathbb{D}_n^L)^2\ +\ (\mathbb{D}_{n-1}^L)^2\ &=[\,(2p-q)+\ \varepsilon(p+2\,q)\,]\,\mathbb{D}_{2n-2}^L+\mathbb{D}_{2n}^L\\&\quad-e_{\mathbb{D}}\ (L_{2n-2}+L_{2n})\,,\end{aligned} \tag{4.4}$$

$$\begin{aligned}(\mathbb{D}_{n+1}^L)^2-(\mathbb{D}_n^L)^2\ &=[\,(2p-q)+\ \varepsilon(p+2\,q)\,]\,\mathbb{D}_{2n+2}^L+\mathbb{D}_{2n-2}^L\\&\quad-e_{\mathbb{D}}\ (L_{2n+2}-L_{2n-2})\,,\end{aligned} \tag{4.5}$$

$$(\mathbb{D}_{n+1}^L)^2+\ e_{\mathbb{D}}\ {L_n}^2=[\,p+\ \varepsilon(p+\ q)\,]\,\mathbb{D}^{\mathbb{L}}{}_{2n+2}+\mathbb{D}_{2n}^L\,, \tag{4.6}$$

$$\mathbb{D}^L_{n-1}\,\mathbb{D}^L_{n+1} - (\mathbb{D}^L_n)^2 = 5\,(-1)^{n+1}\; e_{\mathbb{D}}\,, \tag{4.7}$$

$$\frac{\mathbb{D}^L_{n+r} + (-1)^r\mathbb{D}^L_{n-r}}{\mathbb{D}^L_n} = L_r\ , \tag{4.8}$$

where $e_{\mathbb{D}} = (\,1+\varepsilon)\,e_L$.

Case 2. From properties of the generalized dual Lucas sequence $(\mathbb{D}^L_n)$ for $p=1,\; q=0$ in the equations (4.4) - (4.8), we obtain dual Lucas sequence (D^L_n) as follows:

$$(D^L_n)^2 + \,(D^L_{n-1})^2 = (2+\,\varepsilon)\,D^L_{2n-2} + D^L_{2n} - (1+\,\varepsilon)\;(L_{2n-2} + L_{2n})\,, \tag{4.9}$$

$$(D^L_{n+1})^2 - \,(D^L_n)^2 = (2+\,\varepsilon)\,D^L_{2n+2} + D^L_{2n-2} - (1+\,\varepsilon)\;(L_{2n+2} - L_{2n-2})\,, \tag{4.10}$$

$$(D^L_{n+1})^2 + \,(1+\,\varepsilon)\;{L_n}^2 = (1+\,\varepsilon)\,(D^L_{2n+2} + D^L_{2n})\,, \tag{4.11}$$

$$D^L_{n-1}\,D^L_{n+1} - (D^L_n)^2 = 5\,(-1)^{n+1}\;(1+\,\varepsilon)\,, \tag{4.12}$$

$$\frac{D^L_{n+r} + (-1)^r D^L_{n-r}}{D^L_n} = L_r\ , \tag{4.13}$$

Theorem 4.1 *If $\mathbb{D}^L_n$ is the generalized dual Lucas number, then*

$$\lim_{n\to\infty}\frac{\mathbb{D}^L_{n+1}}{\mathbb{D}^L_n} = \frac{(\,p\,q)\,\alpha^2 + (\,p^2 -\,p\,q +\,q^2\,)\alpha +\,(\,p\,q - q^2)}{q^2\alpha^2 +\,2\,q\,(p-\,q\,)\alpha + (p-q)^2},$$

where $\alpha = 1.618033\cdots$

Proof For the generalized dual Lucas number $\mathbb{D}^L_n$, we obtain

$$\begin{aligned}
\lim_{n\to\infty}\frac{\mathbb{D}^L_{n+1}}{\mathbb{D}^L_n} &= \lim_{n\to\infty}\frac{(p-q+\varepsilon q)L_{n+1} + (q+\varepsilon p)L_{n+2}}{(p-q+\varepsilon q)L_n + (q+\varepsilon p)L_{n+1}} \\
&= \lim_{n\to\infty}\frac{(p^2-pq+q^2)L_nL_{n+1} + (pq-q^2)L_n^2 + pqL_{n+1}^2}{q^2L_{n+1}^2 + 2q(p-q)L_nL_{n+1} + (p-q)^2L_n^2} \\
&\quad + \lim_{n\to\infty}\varepsilon\frac{5(-1)^n(p^2-pq-q^2)}{q^2L_{n+1}^2 + 2q(p-q)L_nL_{n+1} + (p-q)^2L_n^2} \\
&= \frac{(pq)\alpha^2 + (p^2-pq+q^2)\alpha + (pq-q^2)}{q^2\alpha^2 + 2q(p-q)\alpha + (p-q)^2},
\end{aligned} \tag{4.14}$$

where $L_{n+2} = L_{n+1} +\, L_n$.

Case 3. For $p = 1$, $q = 0$ in the equation (4.14), we obtain

$$\lim_{n\to\infty} \frac{\mathbb{D}_{n+1}^L}{\mathbb{D}_n^L} = \lim_{n\to\infty} \frac{D_{n+1}^L}{D_n^L} = \alpha + 0 = \alpha. \qquad \square$$

Theorem 4.2 *The Binet's formula for the generalized dual Lucas sequence is as follows:*

$$\mathbb{D}_n^L = (\tilde{\alpha}\,\alpha^n + \tilde{\beta}\,\beta^n) \tag{4.15}$$

where $\tilde{\alpha} = (p - q + \varepsilon q) + \alpha(q + \varepsilon p)$ *and* $\tilde{\beta} = (p - q + \varepsilon q) + \beta(q + \varepsilon p)$.

Proof If we use definition of the generalized dual Lucas sequence and substitute first equation in footnote, then we get

$$\begin{aligned}
\mathbb{D}_n^L &= (p - q + \varepsilon q)\, L_n + (q + \varepsilon p) L_{n+1} \\
&= (p - q + \varepsilon q)(\alpha^n + \beta^n) + (q + \varepsilon p)(\alpha^{n+1} + \beta^{n+1}) \\
&= \alpha^n (p - q + \varepsilon q + \alpha q + \alpha \varepsilon p) + \beta^n (p - q + \varepsilon q + \beta q + \beta \varepsilon p) \\
&= \tilde{\alpha}\,\alpha^n + \tilde{\beta}\,\beta^n.
\end{aligned} \tag{4.16}$$

where $\tilde{\alpha} = (p - q + \varepsilon q) + \alpha(q + \varepsilon p)$ and $\tilde{\beta} = (p - q + \varepsilon q) + \beta(q + \varepsilon p)$. $\square$

§5. Generalized Dual Lucas Vectors

A generalized dual Lucas vector is defined by

$$\overrightarrow{\mathbb{D}_{\rm n}^{\rm L}} = (\mathbb{D}_n^L, \mathbb{D}_{n+1}^L, \mathbb{D}_{n+2}^L)$$

Also, from equations (4.1), (4.2) and (4.3) it can be expressed as

$$\begin{aligned}
\overrightarrow{\mathbb{D}_{\rm n}^{\rm L}} &= \overrightarrow{\mathbb{L}}_{\rm n} + \varepsilon \overrightarrow{\mathbb{L}_{\rm n+1}} \\
&= (p - q + \varepsilon q)\overrightarrow{\rm L}_n + (q + \varepsilon p)\overrightarrow{\rm L}_{\rm n+1}
\end{aligned} \tag{5.1}$$

where $\overrightarrow{\mathbb{L}_{\rm n}} = (\mathbb{L}_n, \mathbb{L}_{n+1}, \mathbb{L}_{n+2})$ and $\overrightarrow{\rm L}_{\rm n} = (L_n, L_{n+1}, L_{n+2})$ are the generalized Lucas vector and the Lucas vector, respectively.

The product of $\overrightarrow{\mathbb{D}_{\rm n}^{\rm L}}$ and $\lambda \in \mathbb{R}$ is given by

$$\lambda \overrightarrow{\mathbb{D}_{\rm n}^{\rm L}} = \lambda \overrightarrow{\mathbb{L}_{\rm n}} + \varepsilon \lambda \overrightarrow{\mathbb{L}_{\rm n+1}}$$

and $\overrightarrow{\mathbb{D}_{\rm n}^{\rm L}}$ and $\overrightarrow{\mathbb{D}_{\rm m}^{\rm L}}$ are equal if and only if

$$\begin{aligned}
\mathbb{L}_n &= \mathbb{L}_m \\
\mathbb{L}_{n+1} &= \mathbb{L}_{m+1} \\
\mathbb{L}_{n+2} &= \mathbb{L}_{m+2}
\end{aligned}$$

Some examples of the generalized dual Lucas vectors can be given easily as:

$$\begin{aligned}
\overrightarrow{\mathbb{D}_1^{\mathrm{L}}} &= (\mathbb{D}_1^L\,,\,\mathbb{D}_2^L,\,\mathbb{D}_3^L) \\
&= (\mathbb{L}_1\,,\,\mathbb{L}_2,\,\mathbb{L}_3) + \varepsilon(\mathbb{L}_2\,,\,\mathbb{L}_3,\,\mathbb{L}_4) \\
&= (p+2q) + \varepsilon(3p+q), (3p+q) + \varepsilon(4p+3q), (4p+3q) + \varepsilon(7p+4q))
\end{aligned}$$

$$\begin{aligned}
\overrightarrow{\mathbb{D}_2^{\mathrm{L}}} &= (\mathbb{L}_2\,,\,\mathbb{L}_3,\,\mathbb{L}_4) + \varepsilon(\mathbb{L}_3\,,\,\mathbb{L}_4,\,\mathbb{L}_5) \\
&= ((3p+q) + \varepsilon(4p+3q), (4p+3q) + \varepsilon(7p+4q), (7p+4q) + \varepsilon(11p+18q))
\end{aligned}$$

Theorem 5.1 *Let $\overrightarrow{\mathbb{D}_{\mathrm{n}}^{\mathrm{L}}}$ and $\overrightarrow{\mathbb{D}_{\mathrm{m}}^{\mathrm{L}}}$ be two generalized dual Lucas vectors. The dot product of $\overrightarrow{\mathbb{D}_{\mathrm{n}}^{\mathrm{L}}}$ and $\overrightarrow{\mathbb{D}_{\mathrm{m}}^{\mathrm{L}}}$ is given by*

$$\begin{aligned}
\left\langle \overrightarrow{\mathbb{D}_{\mathrm{n}}^{\mathrm{L}}}, \overrightarrow{\mathbb{D}_{\mathrm{m}}^{\mathrm{L}}} \right\rangle = \; & p^2[(L_n\,L_m + 5\,F_{n+m+3}) + \varepsilon(L_n\,L_{m+1} + L_{n+1}\,L_m + 10\,F_{n+m+4})\,] \\
& + p\,q\,[(5\,L_{n+m} + 10\,F_{n+m+2}) \\
& + \varepsilon(L_{n-1}\,L_m + L_n\,L_{m-1} + 10\,F_{n+m-1} + 20\,F_{n+m+3})\,] \\
& + q^2[(L_{n-1}\,L_{m-1} + 5\,F_{n+m+1}) \\
& + \varepsilon(L_{n-1}\,L_m + L_n\,L_{m-1} + 10\,F_{n+m+2})\,]
\end{aligned} \tag{5.2}$$

Proof The dot product of $\overrightarrow{\mathbb{D}_{\mathrm{n}}^{\mathrm{L}}} = (\mathbb{D}_n^L\,,\,\mathbb{D}_{n+1}^L,\,\mathbb{D}_{n+2}^L)$ and $\overrightarrow{\mathbb{D}_{\mathrm{m}}^{\mathrm{L}}} = (\mathbb{D}_m^L\,,\,\mathbb{D}_{m+1}^L,\,\mathbb{D}_{m+2}^L)$ defined by

$$\begin{aligned}
\left\langle \overrightarrow{\mathbb{D}_{\mathrm{n}}^{\mathrm{L}}}, \overrightarrow{\mathbb{D}_{\mathrm{m}}^{\mathrm{L}}} \right\rangle &= \mathbb{D}_n^L\mathbb{D}_m^L + \mathbb{D}_{n+1}^L\mathbb{D}_{m+1}^L + \mathbb{D}_{n+2}^L\mathbb{D}_{m+2}^L \\
&= \left\langle \overrightarrow{\mathbb{L}_{\mathrm{n}}}, \overrightarrow{\mathbb{L}_{\mathrm{m}}} \right\rangle + \varepsilon\,[\,\left\langle \overrightarrow{\mathbb{L}_{\mathrm{n}}}, \overrightarrow{\mathbb{L}_{\mathrm{m+1}}} \right\rangle + \left\langle \overrightarrow{\mathbb{L}_{\mathrm{n+1}}}, \overrightarrow{\mathbb{L}_{\mathrm{m}}} \right\rangle\,]
\end{aligned}$$

where $\overrightarrow{\mathbb{L}_{\mathrm{n}}} = (\,\mathbb{L}_n\,,\,\mathbb{L}_{n+1},\,\mathbb{L}_{n+2})$ is the generalized Lucas vector. Also, the equations (2.1), (2.2) and (2.3), we obtain

$$\begin{aligned}
\left\langle \overrightarrow{\mathbb{L}_{\mathrm{n}}}, \overrightarrow{\mathbb{L}_{\mathrm{m}}} \right\rangle = \; & p^2(L_n\,L_m + 5\,F_{n+m+3}) \\
& + p\,q\,(5\,F_{n+m} + 10\,F_{n+m+2}) \\
& + q^2(L_{n-1}\,L_{m-1} + 5\,F_{n+m+1})
\end{aligned} \tag{5.3}$$

$$\begin{aligned}
\left\langle \overrightarrow{\mathbb{L}_{\mathrm{n}}}, \overrightarrow{\mathbb{L}_{\mathrm{m+1}}} \right\rangle = \; & p^2(L_n\,L_{m+1} + 5\,F_{n+m+4}) \\
& + p\,q\,(5\,F_{n+m-1} + 10\,F_{n+m+3} + L_{n-1}\,L_m) \\
& + q^2(L_{n-1}\,L_m + 5\,F_{n+m+2})\,,
\end{aligned} \tag{5.4}$$

and

$$\begin{aligned}
\left\langle \overrightarrow{\mathbb{L}_{\mathrm{n+1}}}, \overrightarrow{\mathbb{L}_{\mathrm{m}}} \right\rangle = \; & p^2(L_{n+1}\,L_m + 5\,F_{n+m+4}) \\
& + p\,q\,(5\,F_{n+m-1} + 10\,F_{n+m+3} + L_n\,L_{m-1}) \\
& + q^2(L_n\,L_{m-1} + 5\,F_{n+m+2})
\end{aligned} \tag{5.5}$$

Then from equation (5.3), (5.4) and (5.5), we have the equation (5.2). □

Case 1. For the dot product of generalized dual Lucas vectors $\overrightarrow{\mathbb{D}^{\mathrm{L}}_{\mathrm{n}}}$ and $\overrightarrow{\mathbb{D}^{\mathrm{L}}_{\mathrm{n+1}}}$, we get

$$\begin{aligned}
\left\langle \overrightarrow{\mathbb{D}^{\mathrm{L}}_{\mathrm{n}}}, \overrightarrow{\mathbb{D}^{\mathrm{L}}_{\mathrm{n+1}}} \right\rangle &= \mathbb{D}^L_n \mathbb{D}^L_{n+1} + \mathbb{D}^L_{n+1}\mathbb{D}^L_{n+2} + \mathbb{D}^L_{n+2}\mathbb{D}^L_{n+3} \\
&= \left\langle \overrightarrow{\mathbb{L}_{\mathrm{n}}}, \overrightarrow{\mathbb{L}_{\mathrm{n+1}}} \right\rangle + \varepsilon\{ \left\langle \overrightarrow{\mathbb{L}_{\mathrm{n}}}, \overrightarrow{\mathbb{L}_{\mathrm{n+2}}} \right\rangle + \left\langle \overrightarrow{\mathbb{L}_{\mathrm{n+1}}}, \overrightarrow{\mathbb{L}_{\mathrm{n+1}}} \right\rangle \} \\
&= p^2[(L_n\, L_{n+1} + 5\, F_{2n+4}) \\
&+\varepsilon(L_n\, L_{n+2} + L_{n+1}\, L_{n+1} + 10\, F_{2n+5})\,] \\
&+p\, q\,[(5\, L_n\, L_n + L_{n-1}\, L_{n+1} + 10\, F_{2n+3}) \\
&+\varepsilon(L_{n+1}\, L_{n+2} + 5\, F_{2n} + 10\, F_{2n+4})\,] \\
&+q^2[(L_{n-1}\, L_n + 5\, F_{2n+2}) \\
&+\varepsilon(L_{n-1}\, L_{n+1} + L_n\, L_n + 10\, F_{2n+3})\,]
\end{aligned} \tag{5.6}$$

and

$$\begin{aligned}
\left\langle \overrightarrow{\mathbb{D}^{\mathrm{L}}_{\mathrm{n}}}, \overrightarrow{\mathbb{D}^{\mathrm{L}}_{\mathrm{n}}} \right\rangle &= (\mathbb{D}^L_n)^2 + (\mathbb{D}^L_{n+1})^2 + (\mathbb{D}^L_{n+2})^2 \\
&= \left\langle \overrightarrow{\mathbb{L}_{\mathrm{n}}}\,,\ \overrightarrow{\mathbb{L}_{\mathrm{n}}} \right\rangle + 2\,\varepsilon \left\langle \overrightarrow{\mathbb{L}_{\mathrm{n}}}\,,\ \overrightarrow{\mathbb{L}_{\mathrm{n+1}}} \right\rangle \\
&= p^2[(L_n\, L_n + 5\, F_{2n+3}) \\
&+2\,\varepsilon(L_n\, L_{n+1} + 5\, F_{2n+4})\,] \\
&+p\, q\,[(5\, F_{2n} + 10\, F_{2n+2}) \\
&+2\,\varepsilon(L_n\, L_n + L_{n-1}\, L_{n+1} + 10\, F_{2n+3})\,] \\
&+q^2[(L_{n-1}\, L_{n-1} + 5\, F_{2n+1}) \\
&+2\,\varepsilon(L_{n-1}\, L_n + 5\, F_{2n+2})\,]\,.
\end{aligned} \tag{5.7}$$

Then for the norm of the generalized dual Lucas vector [3], we have

$$\begin{aligned}
\left\| \overrightarrow{\mathbb{D}^{\mathrm{L}}_{\mathrm{n}}} \right\| &= \sqrt{\left[\left\langle \overrightarrow{\mathbb{D}^{\mathrm{L}}_{\mathrm{n}}}, \overrightarrow{\mathbb{D}^{\mathrm{L}}_{\mathrm{n}}} \right\rangle \right]} = \sqrt{[(\mathbb{D}^L_n)^2 + (\mathbb{D}^L_{n+1})^2 + (\mathbb{D}^L_{n+2})^2]} \\
&= \sqrt{p^2(L_n\, L_n + 5\, F_{2n+3}) + p\, q(5\, F_{2n} + 10\, F_{2n+2})} \\
&+ \sqrt{q^2(L_{n-1}\, L_{n-1} + 5\, F_{2n+1})]} \\
&+ \sqrt{2\varepsilon\{p^2(L_n\, L_{n+1} + 5\, F_{2n+4}) + pq(L_n\, L_n + L_{n-1}\, L_{n+1} + 10\, F_{2n+3})} \\
&+ \sqrt{q^2(L_{n-1}\, L_n + 5\, F_{2n+2})}\}.
\end{aligned} \tag{5.8}$$

Case 2. For $p = 1$, $q = 0$, in the equations (5.2), (5.6) and (5.8), we have

$$\left\langle \overrightarrow{D^L_{\mathrm{n}}}, \overrightarrow{D^L_{\mathrm{m}}} \right\rangle = [(L_n\, L_m + 5\, F_{n+m+3}) + \varepsilon(L_n\, L_{m+1} + L_{n+1}\, L_m + 10\, F_{n+m+4})\,],$$

---

[3] Norm of dual number as follows ([2], [14]):

$$\left\| \overrightarrow{\mathbb{A}} \right\| = \sqrt{a + \varepsilon\, a^*} = \sqrt{a} + \varepsilon\, a^* \frac{1}{2\sqrt{a}}, A = a + \varepsilon\, a^*$$

$$\left\langle \overrightarrow{D^L_{\mathrm{n}}}, \overrightarrow{D^L_{\mathrm{n+1}}} \right\rangle = [(L_n\,L_{n+1} + 5\,F_{2n+4}) + \varepsilon(L_n\,L_{n+2} + L_{n+1}\,L_{n+1} + 10\,F_{2n+5})\,]$$

and

$$\begin{aligned} \left\| \overrightarrow{D^L_{\mathrm{n}}} \right\| &= \sqrt{(L_n\,L_n + 5\,F_{2n+3}) + 2\,\varepsilon\,(L_n\,L_{n+1} + 5\,F_{2n+4})} \\ &= (L_n\,L_n + 5\,F_{2n+3}) + \varepsilon\,\frac{(L_n\,L_{n+1} + 5\,F_{2n+4})}{\sqrt{(L_n\,L_n + 5\,F_{2n+3})}}. \end{aligned}$$

Theorem 5.2 *Let $\overrightarrow{\mathbb{D}^{\mathrm{L}}_{\mathrm{n}}}$ and $\overrightarrow{\mathbb{D}^{\mathrm{L}}_{\mathrm{m}}}$ be two generalized dual Lucas vectors. The cross product of $\overrightarrow{\mathbb{D}^{\mathrm{L}}_{\mathrm{n}}}$ and $\overrightarrow{\mathbb{D}^{\mathrm{L}}_{\mathrm{m}}}$ is given by*

$$\overrightarrow{\mathbb{D}^{\mathrm{L}}_{\mathrm{n}}} \times \overrightarrow{\mathbb{D}^{\mathrm{L}}_{\mathrm{m}}} = 5\,(-1)^{n+1}\,F_{m-n}\,(1+\varepsilon)\,e_L\;(i+j-k). \tag{5.9}$$

Proof The cross product of $\overrightarrow{\mathbb{D}^{\mathrm{L}}_{\mathrm{n}}} = \overrightarrow{\mathbb{L}_{\mathrm{n}}} + \varepsilon\,\overrightarrow{\mathbb{L}_{\mathrm{n+1}}}$ and $\overrightarrow{\mathbb{D}^{\mathrm{L}}_{\mathrm{m}}} = \overrightarrow{\mathbb{L}_{\mathrm{m}}} + \varepsilon\,\overrightarrow{\mathbb{L}_{\mathrm{m+1}}}$ defined by

$$\overrightarrow{\mathbb{D}^{\mathrm{L}}_{\mathrm{n}}} \times \overrightarrow{\mathbb{D}^{\mathrm{L}}_{\mathrm{m}}} = (\overrightarrow{\mathbb{L}_{\mathrm{n}}} \times \overrightarrow{\mathbb{L}_{\mathrm{m}}}) + \varepsilon\,(\overrightarrow{\mathbb{L}_{\mathrm{n}}} \times \overrightarrow{\mathbb{L}_{\mathrm{m+1}}} + \overrightarrow{\mathbb{L}_{\mathrm{n+1}}} \times \overrightarrow{\mathbb{L}_{\mathrm{m}}})$$

where $\overrightarrow{\mathbb{L}_{\mathrm{n}}}$ is the generalized Lucas vector and $\overrightarrow{\mathbb{L}_{\mathrm{n}}} \times \overrightarrow{\mathbb{L}_{\mathrm{m}}}$ is the cross product for the generalized Lucas vectors $\overrightarrow{\mathbb{L}_{\mathrm{n}}}$ and $\overrightarrow{\mathbb{L}_{\mathrm{m}}}$.

Now, we calculate the cross products $\overrightarrow{\mathbb{L}_{\mathrm{n}}} \times \overrightarrow{\mathbb{L}_{\mathrm{m}}}$, $\overrightarrow{\mathbb{L}_{\mathrm{n}}} \times \overrightarrow{\mathbb{L}_{\mathrm{m+1}}}$ and $\overrightarrow{\mathbb{L}_{\mathrm{n+1}}} \times \overrightarrow{\mathbb{L}_{\mathrm{m}}}$:
Using the property $L_nL_{m+1} - L_{n+1}L_m = 5\,(-1)^n\,F_{m-n}$, we get

$$\overrightarrow{\mathbb{L}_{\mathrm{n}}} \times \overrightarrow{\mathbb{L}_{\mathrm{m}}} = 5\,(-1)^{n+1}\,F_{m-n}\,(i+j-k)\,e_L, \tag{5.10}$$

$$\overrightarrow{\mathbb{L}_{\mathrm{n}}} \times \overrightarrow{\mathbb{L}_{\mathrm{m+1}}} = 5\,(-1)^{n+1}\,F_{m-n+1}(i+j-k)\,e_L, \tag{5.11}$$

and

$$\overrightarrow{\mathbb{L}_{\mathrm{n+1}}} \times \overrightarrow{\mathbb{L}_{\mathrm{m}}} = 5\,(-1)^{n+2}\,F_{m-n-1}(i+j-k)\,e_L. \tag{5.12}$$

Then from the equations (5.10), (5.11) and (5.12), we obtain the equation (5.9). □

Case 3. For $p = 1$, $q = 0$ in the equations (5.9), we have

$$\overrightarrow{D^L_{\mathrm{n}}} \times \overrightarrow{D^L_{\mathrm{m}}} = 5\,(-1)^{n+1}\,F_{m-n}\,(1+\varepsilon)\,(i+j-k).$$

Theorem 5.3 *Let $\overrightarrow{\mathbb{D}^{\mathrm{L}}_{\mathrm{n}}}$, $\overrightarrow{\mathbb{D}^{\mathrm{L}}_{\mathrm{m}}}$ and $\overrightarrow{\mathbb{D}^{\mathrm{L}}_{\mathrm{k}}}$ be the generalized dual Lucas vectors. The mixed product of these vectors is*

$$\left\langle \overrightarrow{\mathbb{D}^{\mathrm{L}}_{\mathrm{n}}} \times \overrightarrow{\mathbb{D}^{\mathrm{L}}_{\mathrm{m}}}\,,\, \overrightarrow{\mathbb{D}^{\mathrm{L}}_{\mathrm{k}}} \right\rangle = 0. \tag{5.13}$$

Proof Using the properties

$$\overrightarrow{\mathbb{D}^{\mathrm{L}}_{\mathrm{n}}} \times \overrightarrow{\mathbb{D}^{\mathrm{L}}_{\mathrm{m}}} = (\,\overrightarrow{\mathbb{L}_{\mathrm{n}}} \times \overrightarrow{\mathbb{L}_{\mathrm{m}}}\,) + \varepsilon\,(\,\overrightarrow{\mathbb{L}_{\mathrm{n}}} \times \overrightarrow{\mathbb{L}_{\mathrm{m+1}}} + \overrightarrow{\mathbb{L}_{\mathrm{n+1}}} \times \overrightarrow{\mathbb{L}_{\mathrm{m}}}\,)$$

and

$$\overrightarrow{\mathbb{D}_{\mathrm{k}}^{\mathrm{L}}} = \overrightarrow{\mathbb{L}_{\mathrm{k}}} + \varepsilon \, \overrightarrow{\mathbb{L}_{\mathrm{k+1}}},$$

we can write,

$$\begin{aligned} \left\langle \overrightarrow{\mathbb{D}_{\mathrm{n}}^{\mathrm{L}}} \times \overrightarrow{\mathbb{D}_{\mathrm{m}}^{\mathrm{L}}},\ \overrightarrow{\mathbb{D}_{\mathrm{k}}^{\mathrm{L}}} \right\rangle &= \left\langle \overrightarrow{\mathbb{L}_{\mathrm{n}}} \times \overrightarrow{\mathbb{L}_{\mathrm{m}}}, \overrightarrow{\mathbb{L}_{\mathrm{k}}} \right\rangle + \varepsilon \, [\left\langle \overrightarrow{\mathbb{L}_{\mathrm{n}}} \times \overrightarrow{\mathbb{L}_{\mathrm{m}}}, \overrightarrow{\mathbb{L}_{\mathrm{k+1}}} \right\rangle \\ &\quad + \left\langle \overrightarrow{\mathbb{L}_{\mathrm{n}}} \times \overrightarrow{\mathbb{L}_{\mathrm{m+1}}}, \overrightarrow{\mathbb{L}_{\mathrm{k}}} \right\rangle + \left\langle \overrightarrow{\mathbb{L}_{\mathrm{n+1}}} \times \overrightarrow{\mathbb{L}_{\mathrm{m}}}, \overrightarrow{\mathbb{L}_{\mathrm{k+1}}} \right\rangle] \,. \end{aligned}$$

Then using equations (5.10), (5.11) and (5.12), we obtain

$$\left\langle (i+j-k),\ \overrightarrow{\mathbb{L}_{\mathrm{k}}} \right\rangle = \mathbb{L}_{\mathrm{k}} + \mathbb{L}_{\mathrm{k+1}} - \mathbb{L}_{\mathrm{k+2}} = 0,$$

$$\left\langle (i+j-k),\ \overrightarrow{\mathbb{L}_{\mathrm{k+1}}} \right\rangle = \mathbb{L}_{\mathrm{k+1}} + \mathbb{L}_{\mathrm{k+2}} - \mathbb{L}_{\mathrm{k+3}} = 0\,.$$

Thus, we have the equation (5.13). □

References

[1] Berzsenyi G., Sums of product of generalized Fibonacci numbers, *The Fibonacci Quarterly*, 13 (4), (1975), 343-344.

[2] Ercan Z. and Yüce S., On properties of the Dual quaternions, *Europen Journal of Pure and Applied Mathematics*, Vol. 4, No.2 (2011) 142-146.

[3] Horadam A. F., A generalized Fibonacci sequence, *American Math. Monthly*, 68, (1961), 455-459.

[4] Horadam A. F., Complex Fibonacci numbers and Fibonacci quaternions, *American Math. Monthly*, 70, (1963), 289-291.

[5] Horadam A. F., Basic properties of a certain generalized sequence of numbers, *The Fibonacci Quarterly*, 3(3), (1965), 161-176.

[6] Iyer M.R., Identities involving generalized Fibonacci numbers, *The Fibonacci Quarterly*, 7(1), (1969), 66-73.

[7] Kalman D. And Mena R., *The Fibonacci Numbers-Exposed*, 76(3), (2003), 167-181.

[8] Rosen K. H., *Discrete Mathematics and its Applications*, McGraw-Hill 1999.

[9] Walton J. E. and Horadam A. F., *Some Aspects of Generalized Fibonacci Numbers*, 12(3), (1974), 241-250.

[10] Güven I. A. and Nurkan S. K., A new approach to Fibonacci, Lucas numbers and dual vectors, *Adv. Appl. Clifford Algebras*, 12(3), (2014), 241-250 .

[11] Vajda S., *Fibonacci and Lucas Numbers the Golden Section*, Ellis Horrowood Limited Publ., England, 1989.

[12] Koshy T., *Fibonacci and Lucas Numbers with Applications*, John Wiley and Sons, Proc., New York-Toronto, 2001.

[13] Yüce S. and Torunbalcı Aydın F., Generalized dual Fibonacci sequence, *The International Journal of Science & Technoledge*, 4(9), (Sep 2016), 193-200.

[14] Agrawal O. P., Hamilton operators and dual number quaternions in spectral kinematik,

Mech. Mach. Theory, Vol.22, no.6, (1987), 569-575.

[15] Siar Z. and Keskin R., The square terms in generalized Lucas sequences, *Mathematika*, 60(1), (Jan 2014), 85-100.

[16] Siar Z. and Keskin R., The square terms in generalized Lucas Sequence with parameters p and q, *Mathematica Scandinavica*, 118(1), (2016), 13-26.

[17] Bilgici G., Two generalizations of Lucas sequence, *Applied Mathematics and Computation*, 245, (Oct 2014), 526-538.